

# Exact Inflationary Models and Consequences

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## Abstract

New exact inflationary solutions are presented in the scalar field theory, minimally coupled to gravity, with a potential term. No use is made of the *slow rollover approximation*. The scale factors are completely nonsingular and the transition from the inflationary phase to the decelerating phase is smooth. Moreover, in one of these models, asymptotically one has transition to the matter dominated Universe.

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# I. Introduction

Inflation was historically introduced to solve the problem of monopoles that could be largely produced in the early Universe. It was subsequently elaborated into other forms such as the *chaotic inflation* scenario [1-3]. Inflation consists of a short period of accelerated superluminal expansion of the early Universe, at the end of which the transition to the standard big bang model should occur [4]. This apparently solves the problem of the flatness and smoothness of the Universe over such a large scale of distances [5-6].

In the scalar field driven inflationary scenario, the matter content of the Universe has the equation of state of the quantum vacuum,  $P = -\rho$  (with  $P, \rho$  the pressure and energy density of matter). The evolution is regarded as the "rolling" of the value of the field which is minimally coupled to gravity, in the presence of the scalar field potential. This potential is motivated usually from particle physics arguments. This is effectively a period of supercooling of the Universe where in fact through a phase transition the field rolls to its true vacuum state [3,5].

The equations of motion for the scale factor of the Universe and the field are usually treated in the so called *slow rollover* approximation. This is effectively equivalent to the flatness of the potential around the "false vacuum" value of the scalar field. However this is not always necessary. Motivated by this we present a class of solutions to these equations that are treated exactly. No use is made of the approximation and the various terms in the scalar field equation of motion are present throughout. These matters are treated in sections II, and III, while in section IV the numerical treatment of the problem is given. These models are nonsingular for all the positive values of the cosmic time and in one of them occurs, in a natural manner, the asymptotic limit of a matter dominated Universe.

## II. Inflation in FLRW Models

In this section we review some basic facts about inflationary models. Then the first exact solution is presented which is generalized in the next section.

The metric is a member of the Friedman-Lemaitre- Robertson-Walker class with flat spatial sections

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \quad (1)$$

where  $a(t)$  is the scale factor of the Universe. Inflation is essentially scalar field dynamics for a scalar field  $\phi$ , minimally coupled to gravity. The energy density and pressure are given by

$$\rho = \frac{(\dot{\phi})^2}{2} + V \quad (2)$$

$$P = \frac{(\dot{\phi})^2}{2} - V, \quad (3)$$

where  $V(\phi)$  is the potential and  $\phi(t)$  is assumed to depend solely on time. The Einstein's equations become

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \quad (4)$$

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}, \quad (5)$$

with the possible presence of the cosmological constant, which will be neglected in what follows. The necessary condition for inflation is  $\ddot{a} > 0$ . This corresponds to a negative pressure satisfying  $P < -\rho/3$ . Also we have the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0. \quad (6)$$

Only two of Eqs. (4)-(6) are truly independent. For example the first follows from the rest. Also the equation for the scalar field is equivalent to the energy conservation equation

$$\dot{\rho} + 3H(\rho + P) = 0. \quad (7)$$

We will consider only equations (5) and (6).

Different inflationary scenarios correspond to different choices of the scalar field potential  $V(\phi)$  motivated by particle physics arguments.

The ansatz for the exact solution is

$$\dot{\phi} = c_0 V^{1/2}(\phi), \quad (8)$$

where for the present purposes  $c_0$  is a constant. Then the solution to Eqs. (5)-(6) is given by

$$\begin{aligned} V(\phi) &= V_0 e^{\lambda\phi} \\ \lambda &\equiv -\frac{c_0 \sqrt{24\pi G}}{[1 + c_0^2/2]^{1/2}} \quad (c_0 > 0) \\ \phi(t) &= -\frac{2}{\lambda} \ln \left[ 1 - \frac{\lambda}{2} c_0 V_0^{1/2} t \right], \quad (\phi(0) = 0) \\ H &= \sqrt{\frac{8\pi G}{3}} \left[ 1 + \frac{c_0^2}{2} \right]^{1/2} V^{1/2}(\phi). \end{aligned} \quad (9)$$

The condition for inflation ( $\ddot{a} > 0$ ) is satisfied, from Eq. (4), for  $c_0^2 < 1$ . The scale factor becomes

$$a(t) = a_0 \left[ 1 - \frac{\lambda}{2} c_0 V_0^{1/2} t \right]^E, \quad (10)$$

where  $a_0 > 0$  and  $E$  is worked out to be  $E = (1/3) + (2/3c_0^2)$ . This is completely nonsingular for all  $t \geq 0$ . The inflationary potential does not specify the equation of

state for the scalar field, which has to be determined. In anticipation of the choice  $0 < c_0 \ll 1$  that will be made, we obtain

$$\frac{P}{\rho} = \frac{(c_0^2/2) - 1}{(c_0^2/2) + 1} \simeq -1, \quad (11)$$

which is the equation of state of the quantum vacuum. The number of e-folds of inflation is given by

$$N(\phi_1 \rightarrow \phi_2) \equiv \int_{t_1}^{t_2} H dt = \sqrt{\frac{8\pi G}{3}} \frac{1}{c_0} \left[ 1 + \frac{c_0^2}{2} \right]^{1/2} (\phi_2 - \phi_1). \quad (12)$$

This can be made arbitrarily large provided that  $c_0 \ll 1$ . We emphasize that there has not been imposed any condition on the *slow roll parameters*

$$\begin{aligned} \epsilon &= \frac{1}{16\pi G} \left( \frac{V'}{V} \right)^2 \\ \eta &= \frac{1}{8\pi G} \frac{V''}{V}, \end{aligned} \quad (13)$$

i.e.  $\epsilon, |\eta| \ll 1$ . Neither the "acceleration term"  $\ddot{\phi}$ , the "force term"  $V' \equiv dV/d\phi$ , nor the "friction term"  $3H\dot{\phi}$  have been neglected. Of course by choosing a suitably small constant  $c_0$  one effectively mimics the conditions of slow rollover approximation but this is done in the context of an exact solution without any sort of approximation.

### III. Generalization of the Solution

The generalization of this solution comes from Eq. (8) when one considers the case  $c_0 = c_0(t)$ . This function is proportional to the "canonical momentum" of the scalar field  $P_\phi = (\partial L / \partial \dot{\phi}) = \dot{\phi}$ . Substituting into Eq. (5) and solving for  $H$ , one can bring Eq. (6) into the form

$$\begin{aligned} \left(1 + \frac{1}{2}c_0^2\right)\dot{F} + \frac{3\kappa}{2} \left[1 + \frac{1}{2}c_0^2\right]^{1/2} c_0 F^2 - \frac{\dot{c}_0}{c_0} F &= 0 \\ F \equiv \dot{\phi}, \quad \kappa^2 \equiv \frac{8\pi G}{3}. \end{aligned} \quad (14)$$

Setting

$$F = [1 + (f/2)c_0^2]^{-1}, \quad (15)$$

with  $f$  a positive constant, one obtains the following equation to be satisfied by  $c_0(t)$ ,

$$\dot{c}_0 = \frac{3\kappa}{2} \frac{c_0^2 [1 + (1/2)c_0^2]^{1/2}}{[1 + (3/2)fc_0^2 + (f/2)c_0^4]}. \quad (16)$$

The ansatz of Eq. (15) essentially amounts to a choice of an equation of state, because this determines the function  $c_0(t)$  and so an equation of state, through the time dependent version of Eq. (11).

The initial condition is  $c_0(t = 0) = c_{0I} > 0$ . Now inflation will last as long as  $c_0^2(t) < 1$ . The function  $c_0(t)$  is *continuous, strictly increasing* (i.e.  $\dot{c}_0 > 0$ ), therefore is invertible in the range of  $t \in [0, +\infty)$  with  $c_{0I} \leq c_0(t) \leq c_{0F}$ . The behaviour of  $c_0(t)$  will be investigated numerically in the next section. The equation of state is given again by the time-dependent version of Eq. (11). This interpolates between the value -1 for the quantum vacuum (in the early stages) and the asymptotic value +1 (when  $c_0(t) \gg 1$ ) for the stiff matter.

The relevant interesting asymptotic limits are

$$\begin{aligned} (i) \quad \dot{c}_0 &= \frac{3\kappa}{2}c_0^2 \quad (c_0 \ll 1) \\ (ii) \quad \dot{c}_0 &= \frac{3\kappa}{f\sqrt{2}c_0} \quad (c_0 \gg 1). \end{aligned} \tag{17}$$

In the first case we obtain ( $0 \leq t < t_0 \equiv (3\kappa c_{0I}/2)^{-1}$ )

$$\begin{aligned} c_0(t) &= \frac{c_{0I}}{1 - 3c_{0I}\kappa t/2} \\ \phi(t) = t &- \frac{\sqrt{2}}{3\kappa} \text{Arctan} \left[ \frac{\sqrt{2}}{c_{0I}} \left( \frac{3}{2}\kappa c_{0I}t - 1 \right) \right] - \frac{\sqrt{2}}{3\kappa} \text{Arctan} \left[ \frac{\sqrt{2}}{c_{0I}} \right] \\ H &= \kappa \left[ \frac{\dot{\phi}^2}{2} + \frac{\dot{\phi}^2}{c_0^2} \right]^{1/2}. \end{aligned} \tag{18}$$

This yields the scale factor which is regular at  $t \simeq 0$  and has the de Sitter form

$$a(t) \simeq a_0 \exp \left( \kappa \sqrt{2} \frac{t}{c_{0I} \sqrt{2 + c_{0I}^2}} \right). \tag{19}$$

The number of e-folds of inflation, with  $c_{0I} = 0.012$ , is as large as  $c_{0I}^{-1}$ .

In the second case we obtain

$$\begin{aligned} c_0^2(t) &= c_{0I}^2 + \frac{3\kappa}{f\sqrt{2}}t \\ H &= \frac{\kappa}{c_0(t)} \frac{\left(1 + \frac{1}{2}c_0^2\right)^{1/2}}{\left(1 + \frac{f}{2}c_0^2\right)}. \end{aligned} \tag{20}$$

This yields a Hubble parameter which depends on time as  $H(2/3)t^{-1}$  and one obtains, for all the members of this class (for every positive value of  $f$ ) the matter dominated Universe at late evolution times

$$a(t) = a_0 t^{2/3}. \tag{21}$$

We summarize for the sake of completeness the method presented here. One integrates Eq. (16) for  $c_0(t)$ , with initial condition  $c_{0I}$ . Then through Eq. (15) one obtains  $\phi = \phi(t)$  which is invertible (through  $F = \dot{\phi} > 0, \forall t \geq 0$ ) for  $t = t(\phi)$ . Therefore from Eq. (8) one obtains, in principle, the functional form of the potential  $V = V(\phi)$ . Finally with the aid of Eq. (2) one obtains the scalar density  $\rho$  and from Eq. (5) one obtains the scale factor  $a = a(t)$ . Thus this is a straightforward procedure to obtain an explicit, time dependent, solution to the field equations Eq. (2) and Eq. (5-6), along with the equation of state, Eq. (11).

The main advantage of these models is that the scale factors are nonsingular at the time  $t = 0$ . The scalar curvature is finite and therefore these are suitable for the Pre-Big-Bang (PBB) scenarios [5,7]. Also the transition from the inflationary to the decelerating era occurs smoothly, when  $c_0(t) > 1$ , at a time controlled by the initial value  $c_{0I}$ . Also in a natural manner, in one of these models the matter dominated Universe occurs in the late stages of the evolution. These aspects are investigated numerically in the next section.

## IV. Numerical Results

In this section we present the results of the numerical treatment of the set of differential equations as they appear in Eqs. (15), (16) and Eq. (18c). The initial conditions are  $0 < c_0(t = 0) = c_{0I} \ll 1$ , while for the scalar field  $\phi(t = 0) = 0$ . That is we use Eq. (16) and

$$\dot{\phi} = \frac{1}{[1 + (f/2)c_0^2(t)]} \quad (22)$$

$$\dot{a} = a \frac{\kappa \dot{\phi}}{c_0} \sqrt{(1 + (1/2)c_0^2(t))}. \quad (23)$$

For the scale factor we use  $a(t = 0) = a_0 > 0$ . We use units where  $\hbar = c = 1$  so that  $t_{Planck} = (\hbar G/c^5) \sim \kappa$ . The results for the scale factors are plotted in Figs. (1)-(2). In these plots the scale factors are normalized to their initial values which are finite, therefore the plots cross the vertical axis at 1, for  $t = 0$ . Finally through the numerical integration of the system, for the range of initial conditions  $0.01 \leq c_{0I} \leq 0.05$ , the dependence of the time that inflation ends,  $t_*$  (in Planck units), on  $c_{0I}$  has the form  $t_* = 0.063 c_{0I}^{-1/2}$ .

## VI. Conclusions and Discussion

In this paper we have presented some new solutions to the scalar field-driven inflation. The main interesting features of these models can be summarized as follows: These are exact solutions and no use of the slow-roll approximation has been made. All the terms in the scalar field equation of motion (Klein-Gordon equation) are present. Also in this class of models the scale factor is nonsingular for all positive values of the evolution time. Moreover the transition from the inflationary to the decelerating phase is smooth and occurs in a cosmic time that depends on the initial value of the canonical momentum of the scalar field. Finally for all the members of this class there exists the asymptotic transition to the matter dominated Universe. The number of e-folds of inflation is high enough to solve the well known problems of the standard model, by a suitable choice of the initial value of the scalar field, although improvement is certainly possible. This is currently under investigation.

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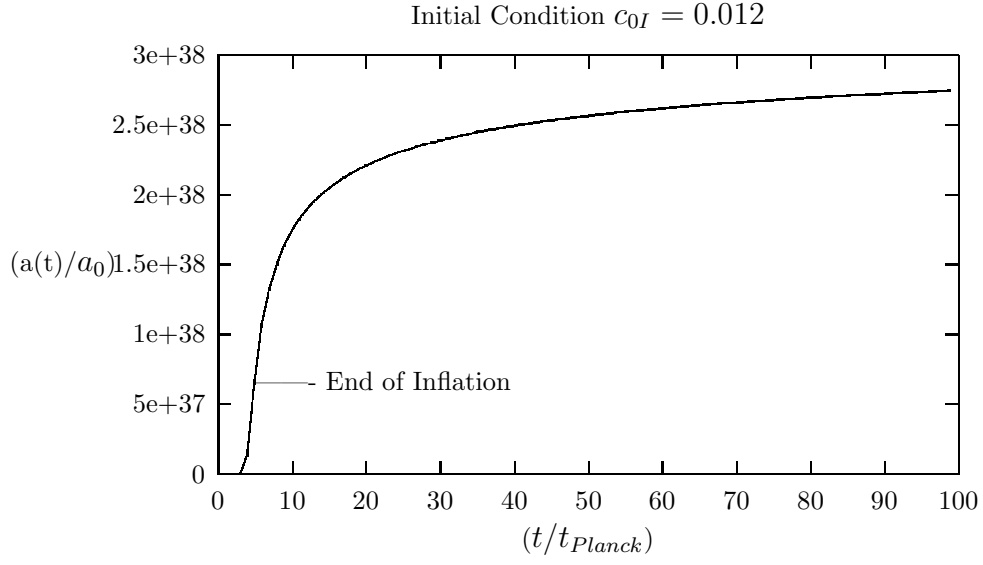


Figure 1: Normalized Scale factor for  $c_{0I} = 0.012$

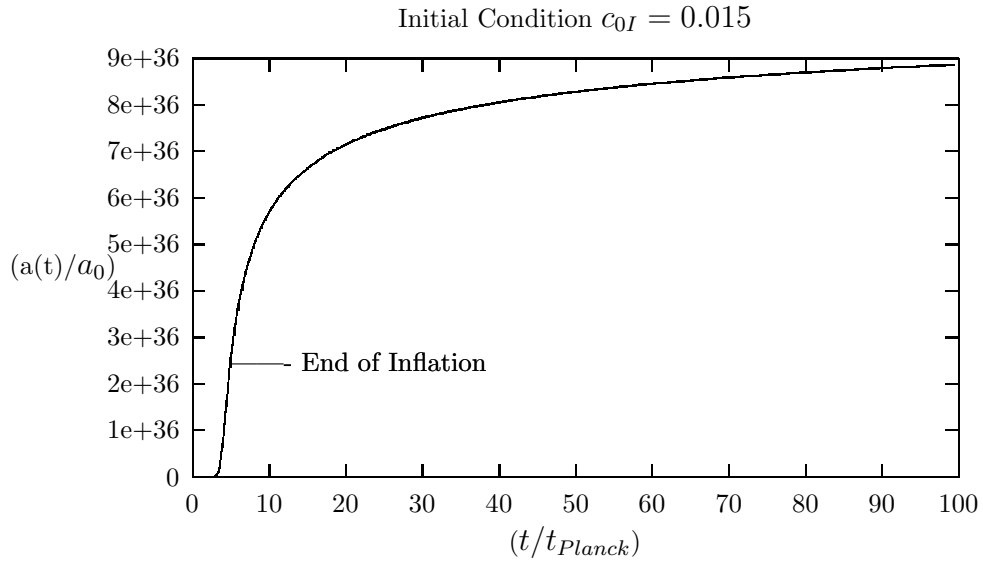


Figure 2: Normalized Scale factor for  $c_{0I} = 0.015$